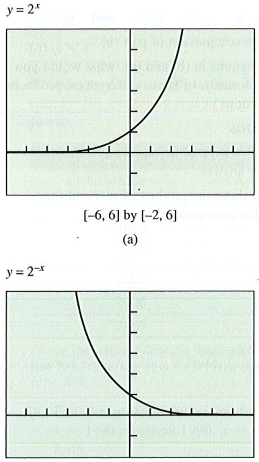
What you will learn about . . .

- Exponential Growth
- Exponential Decay
- Applications
- The Number e

and why . . .

Exponential functions model many growth patterns.


$y = 2^x$

[−6, 6] by [−2, 6]

(a)

$y = 2^{-x}$

[−6, 6] by [−2, 6]

(b)

Figure 1.22 A graph of (a) $y = 2^x$ and (b) $y = 2^{-x}$.

1.3 Exponential Functions

Exponential Growth

Table 1.6 shows the growth of \$100 invested in 1996 at an interest rate of 5.5%, compounded annually.

TABLE 1.6 Savings Account Growth

Year	Amount (dollars)	Increase (dollars)
1996	100	
		5.50
1997	$100(1.055) = 105.50$	
		5.80
1998	$100(1.055)^2 = 111.30$	
		6.12
1999	$100(1.055)^3 = 117.42$	
		6.46
2000	$100(1.055)^4 = 123.88$	

After the first year, the value of the account is always 1.055 times its value in the previous year. After n years, the value is $y = 100 \cdot (1.055)^n$.

Compound interest provides an example of *exponential growth* and is modeled by a function of the form $y = P \cdot a^x$, where P is the initial investment and a is equal to 1 plus the interest rate expressed as a decimal.

The equation $y = P \cdot a^x$, $a > 0$, $a \neq 1$, identifies a family of functions called *exponential functions.* Notice that the ratio of consecutive amounts in Table 1.6 is always the same: $111.30/105.30 = 117.42/111.30 = 123.88/117.42 \approx 1.055$. This fact is an important feature of exponential curves that has widespread application, as we will see.

EXPLORATION 1 Exponential Functions

1. Graph the function $y = a^x$ for $a = 2, 3, 5$, in a $[-5, 5]$ by $[-2, 5]$ viewing window.
2. For what values of x is it true that $2^x < 3^x < 5^x$?
3. For what values of x is it true that $2^x > 3^x > 5^x$?
4. For what values of x is it true that $2^x = 3^x = 5^x$?
5. Graph the function $y = (1/a)^x = a^{-x}$ for $a = 2, 3, 5$.
6. Repeat parts 2–4 for the functions in part 5.

DEFINITION Exponential Function

Let a be a positive real number other than 1. The function

$$f(x) = a^x$$

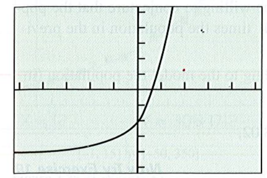
is the **exponential function with base *a*.**

The domain of $f(x) = a^x$ is $(-\infty, \infty)$ and the range is $(0, \infty)$. If $a > 1$, the graph of f looks like the graph of $y = 2^x$ in Figure 1.22a. If $0 < a < 1$, the graph of f looks like the graph of $y = 2^{-x}$ in Figure 1.22b.

EXAMPLE 1 Graphing an Exponential Function

Graph the function $y = 2(3^x) - 4$. State its domain and range.

continued


$y = 2(3^x) - 4$

[−5, 5] by [−5, 5]

Figure 1.23 The graph of $y = 2(3^x) - 4$. (Example 1)

SOLUTION

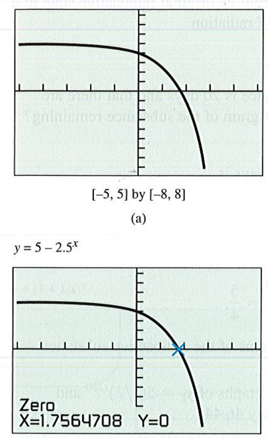
Figure 1.23 shows the graph of the function y. It appears that the domain is $(-\infty, \infty)$. The range is $(-4, \infty)$ because $2(3^x) > 0$ for all x. ***Now Try Exercise 1.***

EXAMPLE 2 Finding Zeros

Find the zeros of $f(x) = 5 - 2.5^x$ graphically.

SOLUTION

Figure 1.24a suggests that f has a zero between $x = 1$ and $x = 2$, closer to 2. We can use our grapher to find that the zero is approximately 1.756 (Figure 1.24b). ***Now Try Exercise 9.***



$y = 5 - 2.5^x$

[−5, 5] by [−8, 8]

(a)

$y = 5 - 2.5^x$

[−5, 5] by [−8, 8]

(b)

Figure 1.24 (a) A graph of $f(x) = 5 - 2.5^x$. (b) Showing the use of the ZERO feature to approximate the zero of f. (Example 2)

Exponential functions obey the rules for exponents.

Rules for Exponents

If $a > 0$ and $b > 0$, the following hold for all real numbers x and y.

1. $a^x \cdot a^y = a^{x+y}$
2. $\dfrac{a^x}{a^y} = a^{x-y}$
3. $(a^x)^y = (a^y)^x = a^{xy}$
4. $a^x \cdot b^x = (ab)^x$
5. $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$

In Table 1.6 we observed that the ratios of the amounts in consecutive years were always the same, namely the interest rate. Population growth can sometimes be modeled with an exponential function, as we see in Table 1.7 and Example 3.

Table 1.7 gives the United States population for several recent years. In this table we have divided the population in one year by the population in the previous year to get an idea of how the population is growing. These ratios are given in the third column.

TABLE 1.7 United States Population

Year	Population (millions)	Ratio
2002	287.9	
		$290.4/287.9 \approx 1.0087$
2003	290.4	
		$293.2/290.4 \approx 1.0096$
2004	293.2	
		$295.9/293.2 \approx 1.0092$
2005	295.9	
		$298.8/295.9 \approx 1.0098$
2006	298.8	
		$301.6/298.8 \approx 1.0094$
2007	301.6	

Source: Statistical Abstract of the United States, 2009.

EXAMPLE 3 Predicting United States Population

Use the data in Table 1.7 and an exponential model to predict the population of the United States in the year 2012.

continued

SOLUTION

Based on the third column of Table 1.7, we might be willing to conjecture that the population of the United States in any year is about 1.01 times the population in the previous year.

If we start with the population in 2002, then according to the model the population (in millions) in 2012 would be about

$$287.9(1.01)^{10} \approx 318.02,$$

or about 318.02 million people. *Now Try Exercise 19.*

Exponential Decay

Exponential functions can also model phenomena that produce a decrease over time, such as happens with radioactive decay. The **half-life** of a radioactive substance is the amount of time it takes for half of the substance to change from its original radioactive state to a nonradioactive state by emitting energy in the form of radiation.

EXAMPLE 4 Modeling Radioactive Decay

Suppose the half-life of a certain radioactive substance is 20 days and that there are 5 grams present initially. When will there be only 1 gram of the substance remaining?

SOLUTION

Model The number of grams remaining after 20 days is

$$5\left(\frac{1}{2}\right) = \frac{5}{2}.$$

The number of grams remaining after 40 days is

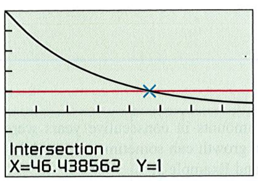
$$5\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = 5\left(\frac{1}{2}\right)^2 = \frac{5}{4}.$$

The function $y = 5(1/2)^{t/20}$ models the mass in grams of the radioactive substance after t days.

Solve Graphically Figure 1.25 shows that the graphs of $y_1 = 5(1/2)^{t/20}$ and $y_2 = 1$ (for 1 gram) intersect when t is approximately 46.44.

Interpret There will be 1 gram of the radioactive substance left after approximately 46.44 days, or about 46 days 10.5 hours. *Now Try Exercise 23.*

$y = 5\left(\frac{1}{2}\right)^{t/20}, y = 1$



[0, 80] by [–3, 5]

Figure 1.25 (Example 4)

TABLE 1.8 U.S. Population

Year	Population (millions)
1880	50.2
1890	63.0
1900	76.2
1910	92.2
1920	106.0
1930	123.2
1940	132.1
1950	151.3
1960	179.3
1970	203.3
1980	226.5
1990	248.7

Source: The Statistical Abstract of the United States, 2004–2005.

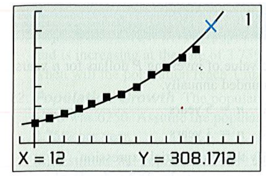
Compound interest investments, population growth, and radioactive decay are all examples of *exponential growth and decay.*

DEFINITIONS Exponential Growth, Exponential Decay

The function $y = k \cdot a^x, k > 0$ is a model for **exponential growth** if $a > 1$, and a model for **exponential decay** if $0 < a < 1$.

Applications

Most graphers have the exponential growth and decay model $y = k \cdot a^x$ built in as an exponential regression equation. We use this feature in Example 5 to analyze the U.S. population from the data in Table 1.8.



[–1, 15] by [–50, 350]

Figure 1.26 (Example 5)

EXAMPLE 5 Predicting the U.S. Population

Use the population data in Table 1.8 to estimate the population for the year 2000. Compare the result with the actual 2000 population of approximately 281.4 million.

SOLUTION

Model Let $x = 0$ represent 1880, $x = 1$ represent 1890, and so on. We enter the data into the grapher and find the exponential regression equation to be

$$f(x) = (56.4696)(1.1519)^x.$$

Figure 1.26 shows the graph of f superimposed on the scatter plot of the data.

Solve Graphically The year 2000 is represented by $x = 12$. Reading from the curve, we find

$$f(12) \approx 308.2.$$

The exponential model estimates the 2000 population to be 308.2 million, an overestimate of approximately 26.8 million, or about 9.5%. *Now Try Exercise 39(a, b).*

EXAMPLE 6 Interpreting Exponential Regression

What *annual* rate of growth can we infer from the exponential regression equation in Example 5?

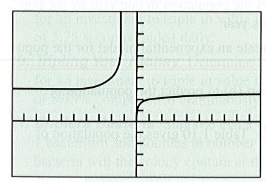
SOLUTION

Let r be the annual rate of growth of the U.S. population, expressed as a decimal. Because the time increments we used were 10-year intervals, we have

$$(1 + r)^{10} \approx 1.1519$$
$$r \approx \sqrt[10]{1.1519} - 1$$
$$r \approx 0.014$$

The annual rate of growth is about 1.4%. *Now Try Exercise 39(c).*

$y = (1 + 1/x)^x$



[–10, 10] by [–5, 10]

X	Y1
1000	2.7169
2000	2.7176
3000	2.7178
4000	2.7179
5000	2.718
6000	2.7181
7000	2.7181

Y1 = (1+1/X)^X

Figure 1.27 A graph and table of values for $f(x) = (1 + 1/x)^x$ both suggest that as $x \to \infty$, $f(x) \to e \approx 2.718$.

The Number *e*

Many natural, physical, and economic phenomena are best modeled by an exponential function whose base is the famous number e, which is 2.718281828 to nine decimal places. We can define e to be the number that the function $f(x) = (1 + 1/x)^x$ approaches as x approaches infinity. The graph and table in Figure 1.27 strongly suggest that such a number exists.

The exponential functions $y = e^x$ and $y = e^{-x}$ are frequently used as models of exponential growth or decay. For example, interest **compounded continuously** uses the model $y = P \cdot e^{rt}$, where P is the initial investment, r is the interest rate as a decimal, and t is time in years.

Quick Review 1.3 (For help, go to Section 1.3.)

Exercise numbers with a gray background indicate problems that the authors have designed to be solved *without a calculator.*

In Exercises 1–3, evaluate the expression. Round your answers to 3 decimal places.

1. $5^{2/3}$ **2.** $3^{\sqrt{2}}$

3. $3^{-1.5}$

In Exercises 4–6, solve the equation. Round your answers to 4 decimal places.

4. $x^3 = 17$ **5.** $x^5 = 24$

6. $x^{10} = 1.4567$

In Exercises 7 and 8, find the value of investing P dollars for n years with the interest rate r compounded annually.

7. $P = \$500$, $r = 4.75\%$, $n = 5$ years

8. $P = \$1000$, $r = 6.3\%$, $n = 3$ years

In Exercises 9 and 10, simplify the exponential expression.

9. $\dfrac{(x^{-3}y^2)^2}{(x^4y^3)^3}$ **10.** $\left(\dfrac{a^3b^{-2}}{c^4}\right)^2\left(\dfrac{a^4c^{-2}}{b^3}\right)^{-1}$

Section 1.3 Exercises

Exercise numbers with a gray background indicate problems that the authors have designed to be solved *without a calculator.*

In Exercises 1–4, graph the function. State its domain and range.

1. $y = -2^x + 3$ **2.** $y = e^x + 3$

3. $y = 3 \cdot e^{-x} - 2$ **4.** $y = -2^{-x} - 1$

In Exercises 5–8, rewrite the exponential expression to have the indicated base.

5. 9^{2x}, base 3 **6.** 16^{3x}, base 2

7. $(1/8)^{2x}$, base 2 **8.** $(1/27)^x$, base 3

In Exercises 9–12, use a graph to find the zeros of the function.

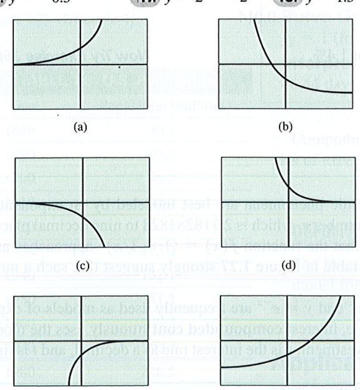
9. $f(x) = 2^x - 5$ **10.** $f(x) = e^x - 4$

11. $f(x) = 3^x - 0.5$ **12.** $f(x) = 3 - 2^x$

In Exercises 13–18, match the function with its graph. Try to do it without using your grapher.

13. $y = 2^x$ **14.** $y = 3^{-x}$ **15.** $y = -3^{-x}$

16. $y = -0.5^{-x}$ **17.** $y = 2^{-x} - 2$ **18.** $y = 1.5^x - 2$



(a) (b) (c) (d) (e) (f)

19. *Population of Nevada* Table 1.9 gives the population of Nevada for several years.

TABLE 1.9 Population of Nevada

Year	Population (thousands)
2002	2168
2003	2238
2004	2330
2005	2409
2006	2492
2007	2565

Source: Statistical Abstract of the United States, 2009.

(a) Compute the ratios of the population in one year by the population in the previous year.

(b) Based on part (a), create an exponential model for the population of Nevada.

(c) Use your model in part (b) to predict the population of Nevada in 2015.

20. *Population of Virginia* Table 1.10 gives the population of Virginia for several years.

TABLE 1.10 Population of Virginia

Year	Population (thousands)
2002	7282
2003	7371
2004	7464
2005	7558
2006	7640
2007	7712

Source: Statistical Abstract of the United States, 2009.

(a) Compute the ratios of the population in one year by the population in the previous year.

(b) Based on part (a), create an exponential model for the population of Virginia.

(c) Use your model in part (b) to predict the population of Virginia in 2012.

In Exercises 21–32, use an exponential model to solve the problem.

21. *Population Growth* The population of Knoxville is 500,000 and is increasing at the rate of 3.75% each year. Approximately when will the population reach 1 million?

22. *Population Growth* The population of Silver Run in the year 1890 was 6250. Assume the population increased at a rate of 2.75% per year.

(a) Estimate the population in 1915 and 1940.

(b) Approximately when did the population reach 50,000?

23. *Radioactive Decay* The half-life of phosphorus-32 is about 14 days. There are 6.6 grams present initially.

(a) Express the amount of phosphorus-32 remaining as a function of time t.

(b) When will there be 1 gram remaining?

24. *Finding Time* If John invests \$2300 in a savings account with a 6% interest rate compounded annually, how long will it take until John's account has a balance of \$4150?

25. *Doubling Your Money* Determine how much time is required for an investment to double in value if interest is earned at the rate of 6.25% compounded annually.

26. *Doubling Your Money* Determine how much time is required for an investment to double in value if interest is earned at the rate of 6.25% compounded monthly.

27. *Doubling Your Money* Determine how much time is required for an investment to double in value if interest is earned at the rate of 6.25% compounded continuously.

28. *Tripling Your Money* Determine how much time is required for an investment to triple in value if interest is earned at the rate of 5.75% compounded annually.

29. *Tripling Your Money* Determine how much time is required for an investment to triple in value if interest is earned at the rate of 5.75% compounded daily.

30. *Tripling Your Money* Determine how much time is required for an investment to triple in value if interest is earned at the rate of 5.75% compounded continuously.

31. *Cholera Bacteria* Suppose that a colony of bacteria starts with 1 bacterium and doubles in number every half hour. How many bacteria will the colony contain at the end of 24 h?

32. *Eliminating a Disease* Suppose that in any given year, the number of cases of a disease is reduced by 20%. If there are 10,000 cases today, how many years will it take

(a) to reduce the number of cases to 1000?

(b) to eliminate the disease; that is, to reduce the number of cases to less than 1?

Group Activity In Exercises 33–36, copy and complete the table for the function.

33. $y = 2x - 3$

x	y	Change (Δy)
1	?	?
2	?	?
3	?	?
4	?	

34. $y = -3x + 4$

x	y	Change (Δy)
1	?	?
2	?	?
3	?	?
4	?	

35. $y = x^2$

x	y	Change (Δy)
1	?	?
2	?	?
3	?	?
4	?	

36. $y = 3e^x$

x	y	Ratio (y_i/y_{i-1})
1	?	?
2	?	?
3	?	?
4	?	

37. Writing to Learn Explain how the change Δy is related to the slopes of the lines in Exercises 33 and 34. If the changes in x are constant for a linear function, what would you conclude about the corresponding changes in y?

38. *Bacteria Growth* The number of bacteria in a petri dish culture after t hours is

$$B = 100e^{0.693t}.$$

(a) What was the initial number of bacteria present?

(b) How many bacteria are present after 6 hours?

(c) Approximately when will the number of bacteria be 200? Estimate the doubling time of the bacteria.

39. *Population of Texas* Table 1.11 gives the population of Texas for several years.

TABLE 1.11 Population of Texas

Year	Population (thousands)
1980	14,229
1990	16,986
1995	18,959
1998	20,158
1999	20,558
2000	20,852

Source: Statistical Abstract of the United States, 2004–2005.

(a) Let $x = 0$ represent 1980, $x = 1$ represent 1981, and so forth. Find an exponential regression for the data, and superimpose its graph on a scatter plot of the data.

(b) Use the exponential regression equation to estimate the population of Texas in 2003. How close is the estimate to the actual population of 22,119,000 in 2003?

(c) Use the exponential regression equation to estimate the annual rate of growth of the population of Texas.

40. *Population of California* Table 1.12 gives the population of California for several years.

TABLE 1.12 Population of California

Year	Population (thousands)
1980	23,668
1990	29,811
1995	31,697
1998	32,988
1999	33,499
2000	33,872

Source: Statistical Abstract of the United States, 2004–2005.

(a) Let $x = 0$ represent 1980, $x = 1$ represent 1981, and so forth. Find an exponential regression for the data, and superimpose its graph on a scatter plot of the data.

(b) Use the exponential regression equation to estimate the population of California in 2003. How close is the estimate to the actual population of 35,484,000 in 2003?

(c) Use the exponential regression equation to estimate the annual rate of growth of the population of California.

Standardized Test Questions

You may use a graphing calculator to solve the following problems.

41. True or False The number 3^{-2} is negative. Justify your answer.

42. True or False If $4^3 = 2^a$, then $a = 6$. Justify your answer.

43. Multiple Choice John invests $200 at 4.5% compounded annually. About how long will it take for John's investment to double in value?

(A) 6 yr (B) 9 yr (C) 12 yr (D) 16 yr (E) 20 yr

44. Multiple Choice Which of the following gives the domain of $y = 2e^{-x} - 3$?

(A) $(-\infty, \infty)$ (B) $[-3, \infty)$ (C) $[-1, \infty)$ (D) $(-\infty, 3]$ (E) $x \neq 0$

45. Multiple Choice Which of the following gives the range of $y = 4 - 2^{2x}$?

(A) $(-\infty, \infty)$ (B) $(-\infty, 4)$ (C) $[-4, \infty)$ (D) $(-\infty, 4]$ (E) all reals

46. Multiple Choice Which of the following gives the best approximation for the zero of $f(x) = 4 - e^x$?

(A) $x = -1.386$ (B) $x = 0.386$ (C) $x = 1.386$ (D) $x = 3$ (E) There are no zeros.

Exploration

47. Let $y_1 = x^2$ and $y_2 = 2^x$.

(a) Graph y_1 and y_2 in $[-5, 5]$ by $[-2, 10]$. How many times do you think the two graphs cross?

(b) Compare the corresponding changes in y_1 and y_2 as x changes from 1 to 2, 2 to 3, and so on. How large must x be for the changes in y_2 to overtake the changes in y_1?

(c) Solve for x: $x^2 = 2^x$.

(d) Solve for x: $x^2 < 2^x$.

Extending the Ideas

In Exercises 48 and 49, assume that the graph of the exponential function $f(x) = k \cdot a^x$ passes through the two points. Find the values of a and k.

48. $(1, 4.5), (-1, 0.5)$

49. $(1, 1.5), (-1, 6)$

Quick Quiz for AP* Preparation: Sections 1.1–1.3

You may use a graphing calculator to solve the following problems.

1. Multiple Choice Which of the following gives an equation for the line through $(3, -1)$ and parallel to the line $y = -2x + 1$?

(A) $= \frac{1}{2}x + \frac{7}{2}$ (B) $y = \frac{1}{2}x - \frac{5}{2}$ (C) $y = -2x + 5$

(D) $y = -2x - 7$ (E) $y = -2x + 1$

2. Multiple Choice If $f(x) = x^2 + 1$ and $g(x) = 2x - 1$, which of the following gives $f \circ g(2)$?

(A) 2 (B) 5 (C) 9 (D) 10 (E) 15

3. Multiple Choice The half-life of a certain radioactive substance is 8 hr. There are 5 grams present initially. Which of the following gives the best approximation when there will be 1 gram remaining?

(A) 2 (B) 10 (C) 15 (D) 16 (E) 19

4. Free Response Let $f(x) = e^{-x} - 2$.

(a) Find the domain of f. **(b)** Find the range of f.

(c) Find the zeros of f.

1.4 Parametric Equations

What you will learn about . . .

- Relations
- Circles
- Ellipses
- Lines and Other Curves

and why . . .

Parametric equations can be used to obtain graphs of relations and functions.

Relations

A **relation** is a set of ordered pairs (x, y) of real numbers. The **graph of a relation** is the set of points in the plane that correspond to the ordered pairs of the relation. If x and y are *functions* of a third variable t, called a *parameter*, then we can use the *parametric mode* of a grapher to obtain a graph of the relation.

EXAMPLE 1 Graphing Half a Parabola

Describe the graph of the relation determined by

$$x = \sqrt{t}, \quad y = t, \quad t \geq 0.$$

Indicate the direction in which the curve is traced. Find a Cartesian equation for a curve that contains the parametrized curve.

SOLUTION

Set $x_1 = \sqrt{t}$, $y_1 = t$, and use the parametric mode of the grapher to draw the graph in Figure 1.28. The graph appears to be the right half of the parabola $y = x^2$. Notice that there is no information about t on the graph itself. The curve appears to be traced to the upper right with starting point $(0, 0)$.

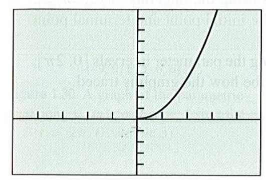
Confirm Algebraically Both x and y will be greater than or equal to zero because $t \geq 0$. Eliminating t we find that for every value of t,

$$y = t = (\sqrt{t})^2 = x^2.$$

Thus, the relation is the function $y = x^2$, $x \geq 0$.

Now Try Exercise 5.

$x = \sqrt{t}, y = t$



$[-5, 5]$ by $[-5, 10]$

Figure 1.28 You must choose a *smallest* and *largest* value for t in parametric mode. Here we used 0 and 10, respectively. (Example 1)

DEFINITIONS Parametric Curve, Parametric Equations

If x and y are given as functions

$$x = f(t), \quad y = g(t)$$

over an interval of t-values, then the set of points $(x, y) = (f(t), g(t))$ defined by these equations is a **parametric curve.** The equations are **parametric equations** for the curve.

The variable t is a **parameter** for the curve and its domain I is the **parameter interval.** If I is a closed interval, $a \leq t \leq b$, the point $(f(a), g(a))$ is the **initial point of the curve** and the point $(f(b), g(b))$ is the **terminal point of the curve.** When we give parametric equations and a parameter interval for a curve, we say that we have **parametrized** the curve. The equations and interval constitute a **parametrization of the curve.**

In Example 1, the parameter interval is $[0, \infty)$, so $(0, 0)$ is the initial point and there is no terminal point.

A grapher can draw a parametrized curve only over a closed interval, so the portion it draws has endpoints even when the curve being graphed does not. Keep this in mind when you graph.